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FLOW BETWEEN CONCENTRIC ROTATION CYLINDERS-- A NOTE ON THE NARROW GAP APPROXIMATION

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Flow between concentric rotating cylinders - - a note on the narrow gap approximation

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Derivations of conditions under which steady flow between concentric rotating cylinders becomes unstable to perturbations of infinitesimal amplitude are often carried out within the framework of the narrow gap approximation in which the separation of the two cylinders is presumed to be much smaller than either radius. Now a narrow gap limit in which the ratio of the separation to a characteristic radius approaches zero is not uniquely defined, but rather the form of the resulting governing equations depends upon the manner in which the passage to the limit is made. A systematic survey of narrow gap limits may therefore be useful.

Let there be a viscous incompressible fluid (density, ρ, kinematic viscosity, ν) confined by concentric cylinders of infinite length (radii, R_1 and R_2). Motion of the fluid is to be caused by rotating the cylinders at angular velocities Ω_1 and Ω_2, by translating them in the direction of their common axis at velocities W_1 and W_2, and by impressing external pressure gradients $P_2' = (\partial p_*/r_*\partial\varphi_*)_e$ and $P_3' = (\partial p_*/\partial z_*)_e$ (where the asterisk denotes the usual, dimensional variable and r_*, φ_*, z_* are cylindrical coordinates of a fixed system whose z_*-axis coincides with the axis of the confining cylinders). We shall choose a characteristic radius R which is of the order R_1 and R_2 (we assume $R_1/R_2 \neq 0$) and define a gap width parameter as

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$$\varepsilon \equiv \frac{d}{R} \quad \text{where} \quad d \equiv R_2 - R_1 \tag{1}$$

A <u>narrow gap limit</u> is then a limit where $\varepsilon \longrightarrow 0$.

Now let us introduce a set of dimensionless variables defined by

$$\left.\begin{aligned} & r_* = R(1+\varepsilon x),\ \varphi_* = \varepsilon^{n_2} y,\ z_* = R\varepsilon^{n_3} z,\ t_* = \frac{R^2}{\nu}\varepsilon^{n_4} t\ , \\ & u_* = \frac{\nu}{R}\,\varepsilon^{m_1} u,\ v_* = \frac{\nu}{R}(\varepsilon^{M_2} V + \varepsilon^{m_2} v),\ w_* = \frac{\nu}{R}(\varepsilon^{M_3} W + \varepsilon^{m_3} w)\ , \\ & p_* = \frac{\rho\nu^2}{R^2}\,(\varepsilon^{M_4} P + \varepsilon^{m_4} p)\ , \quad (n_2 > 0 \text{ and } n_3 > 0) \end{aligned}\right\} \tag{2}$$

where $V(x)$, $W(x)$ and $P(x,y,z)$ describe the equilibrium flow. The equations governing the equilibrium flow are obtained by substituting the relevant parts of equations (2) in the Navier-Stokes equations and the equation of continuity expressed in cylindrical coordinates. Certain terms in the resulting equations are dominated by others when $\varepsilon \longrightarrow 0$ regardless of how M_i and n_i are chosen; when such terms are delted, the governing equations are:

$$\left.\begin{aligned} V^2 &= \varepsilon^{M_4 - 2M_2 - 1} P_x \\ V'' &= \varepsilon^{M_4 + 2 - M_2 - n_2} P_y = \frac{R^3}{\rho\nu^2}\,\varepsilon^{2-M_2} P_2' \\ \text{and} \qquad W'' &= \varepsilon^{M_4 + 2 - M_3 - n_3} P_z = \frac{R^3}{\rho\nu^2}\,\varepsilon^{2-M_3} P_3' \end{aligned}\right\} \tag{3}$$

where subscripts x,y,z denote partial differentiation.

An exponent such as M_4 which can be eliminated from all but one equation can immediately be eliminated altogether; in this case the requirement that V and P shall be present in a narrow gap limit implies that the behavior of P in the limit where $\varepsilon \longrightarrow 0$ is determined by the behavior of V according to the relations,

$$M_4 = 2M_2 + 1 \ , \tag{4}$$

and

$$P_x = V^2$$

Solutions of equations (3) for V and W under the impressed boundary conditions are

$$\left.\begin{aligned} V &= \varepsilon^{-M_2}\left\{\frac{R^2}{\nu}(\bar{\Omega}+x\Delta\Omega)+\frac{1}{2}\,\frac{Rd^2}{\rho\,\nu^2}\,P_2(x-x_1)(x-x_2)\right\} \\ &\text{and} \\ W &= \varepsilon^{-M_3}\left\{\frac{R}{\nu}(\bar{W}+x\Delta W)+\frac{1}{2}\,\frac{Rd^2}{\rho\,\nu^2}\,P_3(x-x_1)(x-x_2)\right\} , \end{aligned}\right\} \tag{5}$$

where x_1 and x_2 are defined by

$$\left.\begin{aligned} & r_*(x_1) = R_1 \quad \text{and} \quad r_*(x_2) = R_2 \ , \\ \text{i.e.} \quad & x_1 = -(R-R_1)/d \quad \text{and} \quad x_2 = (R_2-R)/d \ , \end{aligned}\right\} \tag{6}$$

and where

$$\left.\begin{aligned} \bar{\Omega} &\equiv x_2\Omega_1 - x_1\Omega_2, \quad \Delta\Omega \equiv \Omega_2 - \Omega_1 , \\ \bar{W} &\equiv x_2 W_1 - x_1 W_2, \quad \Delta W \equiv W_2 - W_1 \ . \end{aligned}\right\} \tag{7}$$

The nonlinear equations governing time-dependent perturbations of the steady flow again include terms which always vanish in the limit where $\varepsilon \longrightarrow 0$; after such terms are deleted the equations become

$$
\left.\begin{aligned}
&\varepsilon^{m_1-1} u_x + \varepsilon^{m_2-n_2} v_y + \varepsilon^{m_3-n_3} w_z = 0 \, , \\
&\varepsilon^{m_1} Lu = -\varepsilon^{m_4-1} p_x + (\varepsilon^{m_2} v + \varepsilon^{M_2} V)^2 - 2\varepsilon^{m_2-n_2} v_y \, , \\
&\varepsilon^{m_2} Lv + \varepsilon^{M_2+m_1-1} V' u = -\varepsilon^{m_4-n_2} p_y + 2\varepsilon^{m_1-n_2} u_y
\end{aligned}\right\} \quad (8)
$$

and

$$
\varepsilon^{m_3} Lw + \varepsilon^{M_3+m_1-1} W' u = -\varepsilon^{m_4-n_3} p_z
$$

where the prime denotes $\frac{d}{dx}$,

$$
\left.\begin{aligned}
&L \equiv \varepsilon^{-n_4}\frac{\partial}{\partial t} + \varepsilon^{M_2-n_2} V \frac{\partial}{\partial y} + \varepsilon^{M_3-n_3} W \frac{\partial}{\partial z} + \underline{u}\cdot\underline{\nabla} - \nabla^2 , \\
&\underline{u}\cdot\underline{\nabla} \equiv \varepsilon^{m_1-1} u \frac{\partial}{\partial x} + \varepsilon^{m_2-n_2} v \frac{\partial}{\partial y} + \varepsilon^{m_3-n_3} w \frac{\partial}{\partial z} \, ,
\end{aligned}\right\} \quad (9)
$$

and

$$
\nabla^2 \equiv \varepsilon^{-2}\frac{\partial^2}{\partial x^2} + \varepsilon^{-2n_2}\frac{\partial^2}{\partial y^2} + \varepsilon^{-2n_3}\frac{\partial^2}{\partial z^2}
$$

The linearized equations governing an infinitesimal perturbation of the form

$$
\begin{aligned}
&\underline{u} = \underline{u}(x)\exp\left\{ i(k_2 y + k_3 z) + \lambda t \right\} , \\
&p = p(x)\exp\left\{ i(k_2 y + k_3 z) + \lambda t \right\} ,
\end{aligned} \quad (10)
$$

are

$$\left.\begin{aligned}
&\varepsilon^{m_1-1}Du+i(\varepsilon^{m_2-n_2}k_2v+\varepsilon^{m_3-n_3}k_3w)=0\ ,\\
&Lu=-\varepsilon^{m_4-m_1-1}Dp+2\,\varepsilon^{M_2+m_2-m_1}Vv-2i\,\varepsilon^{m_2-m_1-n_2}k_2v,\\
&Lv=-i\varepsilon^{m_4-m_2-n_2}k_2p-\varepsilon^{M_2+m_1-m_2-1}V'u+2i\,\varepsilon^{m_1-m_2-n_2}k_2u,\\
&\text{and}\\
&Lw=-i\,\varepsilon^{m_4-m_3-n_3}k_3p-\varepsilon^{M_3+m_1-m_3-1}W'u,
\end{aligned}\right\}\qquad(11)$$

where $D \equiv \frac{d}{dx}$

and

$$\left.\begin{aligned}
L \equiv\ &\varepsilon^{-n_4}\lambda+i(\varepsilon^{M_2-n_2}k_2V+\varepsilon^{M_3-n_3}k_3W)+\\
&+(\varepsilon^{-2n_2}k_2^2+\varepsilon^{-2n_3}k_3^2-\varepsilon^{-2}D^2)
\end{aligned}\right\}\qquad(12)$$

In addition to equations (11) the velocity components must satisfy the boundary conditions

$$u=v=w=0 \quad \text{at} \quad x=x_1 \quad \text{and} \quad x=x_2 \qquad (13)$$

With regard to choice of the exponents m_1-1, m_2-n_2, and m_3-n_3, we see immediately that at least two of them must be equal, for, otherwise, the equation of continuity implies that the only solution is the trivial one where $u=v=w=p=0$. The case where two of the above exponents are equal and greater than the third simply cannot occur, for then the equation of continuity implies that the veolcity component associated with the third exponent is identically zero; this means, in effect, that we have tried to chose the exponent m_i associated with the

third component <u>too small</u>. Now the exponents m_i indicate the behavior of the leading terms in expansions of the pressure and the velocity components in powers of ε, and consequently a value for m_i which is too small in the above sense is not allowed since the leading term of an expansion is by universal agreement <u>not</u> identically zero. Thus a rough classification of possible limits on the basis of the continuity equation contains four cases: the case $m_1-1=m_2-n_2=m_3-n_3$ and the three cases where two of the exponents are equal and less than the third. We shall first treat the case where the three exponents are equal; then it will be shown that the remaining cases do not occur.

If we now take $m_1-1=m_2-n_2=m_3-n_3=\alpha$ (say), equations (11) become

$$\left.\begin{aligned} &Du + i(k_2v+k_3w) = 0, \\ &Lu = -\varepsilon^{m_4-\alpha-2}Dp + 2\,\varepsilon^{M_2+n_2-1}Vv, \\ &Lv = -i\,\varepsilon^{m_4-\alpha-2n_2}k_2p - \varepsilon^{M_2-n_2}V'u\,, \\ \text{and}\qquad &Lw = -i\,\varepsilon^{m_4-\alpha-2n_3}k_3p - \varepsilon^{M_3-n_3}W'u\,, \end{aligned}\right\} \qquad (14)$$

where two terms have been dropped since $m_1-m_2-n_2=-2n_2+1$ and $m_2-m_1-n_2=-1$. As we shall see presently, the exponent α, which now remains unspecified, can be assigned if one considers secondary motions of finite amplitude.

The exponents we have to deal with are M_2-n_2, M_3-n_3, M_2+n_2-1, $-2n_2$, $-2n_3$, $-n_4$, and $m_4-\alpha$. Of these, the last two are determined by the others in virtue of the following considerations: (1) $-n_4$ appears but once (in L) and therefore must be set equal to the smallest of the remaining exponents in equations (14) in order that the parameter λ shall appear in the limit and (2) $m_4-\alpha$ appears only in conjunction with p and therefore, in order to insure that p is the leading term of an expansion in powers of ε, must be given the minimum value consistent with the requirement that p shall not vanish identically. Furthermore, we must add the restriction that none of the exponents which appear in L shall be less than -2; otherwise, equations (14) can be reduced to a single second-order differential equation for u which can be shown to have no nontrivial solution which satisfies the boundary conditions u=Du=0 at x_1 and x_2. Thus we have

$$n_2 \leq 1\ ,\quad n_3 \leq 1\ ,\quad M_2 \geq n_2-2,\quad M_3 \geq n_3-2,\quad n_4=2. \tag{15}$$

If we now add to the above restrictions the requirement that the functions V and W shall in fact appear in the limiting equations, we obtain the relations,

$$\left.\begin{array}{lll} & M_3-n_3 = -2 & \text{with} \quad n_3 \leq 1 \\ \text{and} & & \\ & M_2-n_2 = -2 & \text{with} \quad \tfrac{1}{2} \leq n_2 \leq 1 \\ \text{or} & & \\ & M_2+n_2 = -1 & \text{with} \quad n_2 \leq \tfrac{1}{2}\ , \end{array}\right\} \tag{16}$$

from which it follows that a choice of n_2 and n_3 completely determines the remaining exponents.

Now let us consider the case where $n_2=n_3=1$. If we make the transformation

$$\begin{aligned} w &\rightarrow w/k_3 \,, \quad v \rightarrow v/k_2 \,, \\ W &\rightarrow W/k_3 \,, \quad V \rightarrow V/k_2 \,, \end{aligned} \tag{17}$$

we obtain the equations

$$\left.\begin{aligned} &Du + i(v+w) = 0 \\ &Lu = - Dp \\ &Lv = -ik_2^2 p - V'u \\ &Lw = -ik_3^2 p - W'u \end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned} V &= \varepsilon k_2 \left\{\frac{R}{\nu}^2 (\overline{\Omega}+x\Delta\Omega) + \frac{1}{2}\,\frac{Rd^2}{\rho\,\nu^2} P_2 (x-x_1)(x-x_2)\right\} \\ W &= \varepsilon k_3 \left\{\frac{R}{\nu}(\overline{W}+x\Delta\, W) + \frac{1}{2}\,\frac{Rd^2}{\rho\,\nu^2} P_3 (x-x_1)(x-x_2)\right\} \\ L &= \lambda + i(V+W) + (k_2^2+k_3^2-D^2) \end{aligned}\right\} \tag{19}$$

The pressure and two velocity components may be eliminated from equations (18) to give the equation

$$\left\{L(k_2^2+k_3^2-D^2) + i(V''+W'')\right\} u = 0 \tag{20}$$

with the boundary conditions

$$u = Du = 0 \quad \text{at} \quad x = x_1 \ \text{and} \ x = x_2 \tag{21}$$

If we now consider the case where $n_3 < 1$ we obtain governing equations which have the same form as equations (18) through (21) except for a replacement of the quantity k_3 by $k_3 \ \varepsilon^{1-n_3}$ throughout. Thus it is clear that no new narrow gap limits can be obtained by taking $n_3 < 1$ since they are all covered under the case $n_3=1$, $k_3=0$. By means of a similar argument we can establish that the cases where $\frac{1}{2} < n_2 < 1$ are equivalent to the case where $n_2=1$, $k_2=0$.

When $n_2=\frac{1}{2}$ a new term is introduced in the governing equation for u, and a second narrow gap limit is obtained. The governing equations then become

$$\left.\begin{aligned} Du + i(v+w) &= 0 \\ Lu &= -Dp + 2Vv/k_2^2 \\ Lv &= \qquad -V'u \\ Lw &= -ik_3^2 p - W'u \end{aligned}\right\} \tag{22}$$

where

$$\left.\begin{aligned} V &= \varepsilon^{\frac{3}{2}} k_2 \left\{\frac{R^2}{\nu}(\bar{\Omega}+x\Delta\Omega) + \frac{1}{2}\frac{Rd^2}{\rho\nu^2} P_2(x-x_1)(x-x_2)\right\} \\ W &= \varepsilon k_3 \left\{\frac{R}{\nu}(\bar{W}+x\Delta W) + \frac{1}{2}\frac{Rd^2}{\rho\nu^2} P_3(x-x_1)(x-x_2)\right\} \\ L &= \lambda + i(V+W) + (k_3^2 - D^2) \end{aligned}\right\} \tag{23}$$

The pressure and a velocity component may be eliminated from equations (22) to give the equations

$$\{L(k_3^2-D^2) + i(V''+W'')\}\, u = 2\left(\frac{k_3}{k_2}\right)^2 Vu$$

$$Lv = -V'u \tag{24}$$

with the boundary conditions

$$u = Du = v = 0 \quad \text{at} \quad x=x_1 \quad \text{and} \quad x=x_2 \tag{25}$$

When $n_2 < \frac{1}{2}$, the exponent M_2-n_2 must be greater than -2 and we then obtain the equation

$$(\lambda + iW + (k_3^2-D^2))v = 0 \tag{26}$$

From the equation above, its complex conjugate and the boundary conditions $v(x_1)=v(x_2)=0$, we obtain the relation,

$$\lambda+\lambda^* = -\int_{x_1}^{x_2} (k_3^2|v|^2 + |Dv|^2)dx \Big/ \int_{x_1}^{x_2} |v|^2 dx, \tag{27}$$

from which it follows that there can be no unstable solution unless $v \equiv 0$. Thus we find only the two limits given above if we require that all three velocity components shall have nontrivial expansions in powers of ε with leading terms which appear in the limiting form of the continuity equation.

The exponent α which has been left unspecified may be assigned by requiring that equations (8) shall include nonlinear terms which limit the exponential growth of unstable perturbations predicted by the stability equations. It is

readily verified that $\alpha = -2$ is the only choice possible.

Narrow gap limits in which one of the velocity components does not appear in the lowest order continuity equation fall into two categories: (1) limits which can be obtained from the two limits we have discussed above by setting one of the velocity components to zero and (2) limits which introduce new terms in the governing equations. It can be shown that there are no nontrivial solutions of the equations governing the limits in the first category, and thus we need consider only the second.

From equations (11) it follows that no new terms can be introduced into the governing equations when

$$(m_3-n_3) = \beta > (m_2-n_2) = (m_1-n_1) = -2; \tag{28}$$

hence we need not consider this case.

When

$$(m_1-1) = \beta > (m_2-n_2) = (m_3-n_3) = -2 , \tag{29}$$

we have

$$\left.\begin{aligned} M_2+m_1-m_2-1 &= M_2-n_2+(\beta+2) > M_2-n_2 \\ M_3+m_1-m_3-1 &= M_3-n_3+(\beta+2) > M_3-n_3 \\ m_1-m_2-n_2 &= (\beta+2)+1-2n_2 > -2n_2 \end{aligned}\right\} \tag{30}$$

In virtue of equations (29) and (30) the governing equations are reduced to

$$
\left.\begin{aligned}
i(k_2v + k_3w) &= 0\ , \\
Lv &= -i\varepsilon^{m_4-m_2-n_2}k_2p \\
\text{and}\qquad Lw &= i^{\,m_4-m_3-n_3}k_3p\ ,
\end{aligned}\right\} \tag{31}
$$

where L is defined as in equation (12). From the above equations it follows that

$$
(\varepsilon^{m_4-m_2-n_2}k_2^2 + \varepsilon^{m_4-m_3-n_3}k_3^2)p = 0 \tag{32}
$$

and hence that one or both of the equations

$$
Lv = 0 \quad \text{and} \quad Lw = 0 \tag{33}
$$

is satisfied. In any case an equation similar to equation (27) can be derived, and thus we find no unstable solutions.

Finally we have the situation where

$$
m_2-n_2 = \beta > m_1-1 = m_3-n_3 = -2. \tag{34}
$$

In this case we may take

$$
m_1 = m_3 = M_3 = -n_3 = -1\ ,\ n_4 = 2. \tag{35}
$$

In virtue of the relations,

$$
M_2+m_1-m_2-1 < M_2-n_2 \quad \text{and} \quad m_2-m_1-n_2 > -1, \tag{36}
$$

the governing equations become

$$\left.\begin{aligned}
&Du + ik_3w = 0\\
&Lu = \varepsilon^{m_4}Dp+2\,\varepsilon^{M_2+m_2+1}Vv\\
&Lv = i\varepsilon^{m_4-m_2-n_2}k_2p-\varepsilon^{M_2-m_2-2}V'u+2i\,\varepsilon^{-m_2-n_2-1}k_2u\\
&\text{and}\\
&Lw = i\varepsilon^{m_4}k_3p - \varepsilon^{-2}W'u\\
&\text{where}\\
&L = \varepsilon^{-2}\left[(\lambda+ik_3W)+(k_3^2-D^2)\right]+\varepsilon^{-2n_2}k_2^2
\end{aligned}\right\} \qquad (37)$$

If the above system is not to degenerate to a lower order system (for which the boundary conditions cannot all be satisfied), the remaining exponents must be chosen in such manner that no power of ε appearing in the above equations is less than -2. Furthermore, if V is to affect the limit at all we have

$$M_2+m_2+1 = -2\ , \qquad (38)$$

and if the third equation is not to become $Lv = 0$ we have either

$$\left.\begin{aligned}
&m_4-m_2-n_2 = -2\ , &&\text{(a)}\\
&M_1-m_2-2 = -2\ , &&\text{(b)}\\
\text{or}\quad &-m_2-n_2-1 = -2\ . &&\text{(c)}
\end{aligned}\right\} \qquad (39)$$

It can be shown that equations (38) and (39) (a,b, or c) cannot be satisfied when the indices satisfy the inequalities implied in equations (37) and (34). Thus there is no narrow gap limit here, and this exhausts the possible cases.

The two narrow gap limits are:

Limit 1: $n_2=n_3=1,\ n_4=2,\ m_4=-2,$

$$m_2=m_3=M_2=M_3=-1 \quad \text{and} \quad M_4=-1.$$

The governing equations are

$$\left.\begin{aligned}
&\{L(k_2^2+k_3^2-D^2)+i(k_2V''+k_3W'')\}u = 0 \\
&\text{and} \\
&u = Du = 0 \quad \text{at} \quad x=x_1 \quad \text{and} \quad x=x_2 \\
&\text{where} \\
&L = \lambda + i(k_2V+k_3W) + (k_2^2+k_3^2-D^2)
\end{aligned}\right\} \qquad (40)$$

There are the following six Reynolds numbers to be considered in general:

$$\frac{R\bar{\Omega}d}{\nu},\ \frac{R\Delta\Omega d}{\nu},\ \frac{1}{2}\frac{d^3}{\rho\nu^2}P_2,\ \frac{\bar{W}d}{\nu},\ \frac{\Delta Wd}{\nu} \ \text{and}\ \frac{1}{2}\frac{d^3}{\rho\nu^2}P_3 \qquad (41)$$

The first and fourth of the above may be eliminated by a redefinition of λ.

Limit 2: $n_2=\frac{1}{2},\ n_3=1,\ n_4=2,\ m_4=-2,\ m_2=M_2=-\frac{3}{2},$

$$m_3=M_3=-1 \quad \text{and} \quad M_4=-2.$$

The governing equations are

$$\{L(k_3^2-D^2) + i(k_2V'' + k_3W'')\}u = 2k_3^2Vv,$$

$$Lv = -V'u.$$

and

$$u=v=Du=0 \quad \text{at} \quad x=x_1 \quad \text{and} \quad x=x_2 \tag{42}$$

where

$$L = \lambda + i(k_2V+k_3W)+(k_3^2-D^2)$$

Here the six Reynolds numbers are:

$$\frac{R\bar{\Omega}d}{\nu}\sqrt{\frac{d}{R}}, \ \frac{R\Delta\Omega d}{\nu}\sqrt{\frac{d}{R}}, \ \frac{1}{2}\frac{d^3}{\rho\nu^2}P_2\sqrt{\frac{d}{R}}, \ \frac{\bar{W}d}{\nu}, \ \frac{\Delta Wd}{\nu} \text{ and } \frac{1}{2}\frac{d^3}{\rho\nu^2}P_3 \tag{43}$$

In view of the fact that the Reynolds numbers associated with V approach zero for fixed $\bar{\Omega}$, $\Delta\Omega$ and P_2 faster in limit 2 than in limit 1 by the factor $\sqrt{d/R}$, it might be surmised that limit 1 is the more important of the two. In fact this would be exactly the wrong conclusion. Let us consider, in order, the three cases where W vanishes and the dominant term in V is due either to $\bar{\Omega}$, $\Delta\Omega$ or P_2. In the first case the fact that the Reynolds number associated with $\bar{\Omega}$ can be eliminated in limit 1, but not in limit 2, indicates that it is only limit 2 which has any bearing on the problem. In the second case, the neglect of $\frac{1}{2}\,Rd^2P_2/\rho\nu^2$ relative to $R^2\Delta\Omega/\nu$ in limit 1 gives the problem of pure shear flow between parallel planes. Since this problem is known to have only

stable solutions, it is again limit 2 which matters. It is only in the third case that limit 1 actually prevails; however, as Lin has already remarked*, if one adopts the narrow gap approximation, wherein the equations of the narrow gap limit are used for finite but small ε, then on the basis of the computed critical Reynolds numbers for the two limits it is found that limit 1 is dominant only when ε is less than 0.26×10^{-4}; and this is indeed a very narrow gap.

*C. C. Lin <u>Theory of Hydrodynamic Stability</u> p. 48.